**MAS224, Actuarial Mathematics: Normal approximation**

We will need a fact from Probability.

*Central Limit Theorem.* Let $Z_j$, $j = 1, 2, \ldots$, be a sequence of mutually independent and identically distributed random variables each having mean $\mu$ and variance $\sigma^2$. Then the distribution of

$$\frac{Z_1 + \ldots + Z_N - N\mu}{\sigma\sqrt{N}}$$

tends to the standard normal distribution as $N \to \infty$. That is

$$\lim_{N\to\infty} P\left(\frac{Z_1 + \ldots + Z_N - N\mu}{\sigma\sqrt{N}} \leq b\right) = \Phi(b), \qquad \Phi(x) = \int_{-\infty}^{x} \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} du.$$

It follows from the Central Limit Theorem that if $Y$ is the sum of $N$ independent and identically distributed random variables, $Y = Z_1 + \ldots + Z_N$, then

$$P\left(\frac{Y - E(Y)}{\sqrt{\operatorname{var}(Y)}} \leq b\right) \approx \Phi(b) \quad \text{for large } N.$$

Consider now the case when $N$ identical policies are sold by a life insurance office. Let $Z_j$ be the present value of policy $j$, $j = 1, \ldots, N$, and $Y$ be the total present value of this block of policies, $Y = \sum_{j=1}^{N} Z_j$. The random variables $Z_j$ are mutually independent since they relate to different individuals. Then the Central Linit Theorem can be used to see how much money, £$C$, needs to be invested by the life company in order to have probability $1 - \alpha$ of generating funds to pay the benefit payments. Indeed, $C$ is determined by the condition

$$P(Y \leq C) = 1 - \alpha.$$

The event $Z \leq C$ is equivalent to the event $Y - E(Y) \leq C - E(Y)$, and the latter is equivalent to the event $[Y - E(Y)]/\sqrt{\operatorname{var}(Y)} \leq [C - E(Y)]/\sqrt{\operatorname{var}(Y)}$, so that

$$P(Y \leq C) \leq= P\left(\frac{Y - E(Y)}{\sqrt{\operatorname{var}(Y)}} \leq \frac{C - E(Y)}{\sqrt{\operatorname{var}(Y)}}\right).$$

Therefore, the above equation for $C$ becomes

$$P\left(\frac{Y - E(Y)}{\sqrt{\operatorname{var}(Y)}} \leq \frac{C - E(Y)}{\sqrt{\operatorname{var}(Y)}}\right) = 1 - \alpha.$$

By the Central Limit Theorem, for large $N$, the probability distribution of $\frac{Y - E(Y)}{\sqrt{\operatorname{var}(Y)}}$ can be approximated by $N(0, 1)$,

$$P\left(\frac{Y - E(Y)}{\sqrt{\operatorname{var}(Y)}} \leq \frac{C - E(Y)}{\sqrt{\operatorname{var}(Y)}}\right) \approx \Phi\left(\frac{C - E(Y)}{\sqrt{\operatorname{var}(Y)}}\right).$$

Hence $C$ can be determined from the equation

$$\Phi\left(\frac{C - E(Y)}{\sqrt{\operatorname{var}(Y)}}\right) = 1 - \alpha.$$

Let $z_\alpha$ be the upper $100\alpha$ percentage point of the standard normal distribution, so that $\Phi(z_\alpha) = 1 - \alpha$. Then, from the above equation for $C$,

$$\frac{C - E(Y)}{\sqrt{\operatorname{var}(Y)}} = z_\alpha, \quad \text{or, equivalently,} \quad C = E(Y) + z_\alpha\sqrt{\operatorname{var}(Y)}.$$

Recall now that $Z$ is the total present value of $N$ policies, $Y = \sum_{j=1}^{N} Z_j$. If $\mu$ and $\sigma^2$ are correspondingly the expected value and variance of each individual policy, ($\mu = E(Z_j)$ and $\sigma^2 = \operatorname{var}(Z_j)$) then $E(Y) = N\mu$ and $\operatorname{var} Y = N\sigma^2$. Thus, finally,

$$C = N\mu + z_\alpha\sqrt{N}\sigma.$$

Note that the amount that needs to be invested per policy is $\mu + z_\alpha\sigma/\sqrt{N}$.

*Example.* Consider a pure endowment policy for 5000 taken out for a child on his fifth birthday which matures on survival to his 21st birthday. Assume that his mortality is that of ELT12 and the rate of interest is 13% per annum. Calculate the expectation and standard deviation of the present value of the benefit.

A life assurance company withdraws benefit payments from a fund earning interest at rate 13% per annum. They have just sold a block of 100 endowment policies identical to the one above. Let Xj be the present value of the jth policy. Calculate the minimum amount x per policy that the company needs to invest in the fund now so that there is a probability of (approximately) 0.95 that the total amount invested will be sufficient to pay the benefit payments incurred by these 100 endowment policies.

The expected present value of the n-year pure endowment policy with unit benefit is given by $(v^n)_n p_x$ and the variance is given by $v^{2n}{}_n p_{xn} q_x$. Therefore, the mean $\mu$ and variance $\sigma^2$ of the present value of the benefit are just

$$\mu = 5000 v^{16}\frac{l_{21}}{l_5} = \frac{5000 \times 96178}{(1.13)^{16} \times 97175} = 700.222564$$

and

$$\sigma^2 = \frac{(5000)^2(l_{21})(l_5 - l_{21})}{(1.13)^{32}(l_5)} = \frac{(5000)^2(96178)(97175 - 96178)}{(1.13)^{32}(97175)^2} = 5082.666551$$

So $\sigma = \sqrt{\sigma^2} = 71.292823$

The upper 95% point of the standard normal distribution is 1.6449. Therefore the minimum amount the company needs to invest so that there is a probability of (approximately) 0.95 that this will be sufficient to pay the benefit payments incurred by these 100 endowment policies is

$$\mu + 1.6449 \times \sigma/\sqrt{100} = 700.222564 + \frac{1.6449 \times 71.292823}{10} = 711.9495$$

So the minimum amount the company needs to invest per policy is $\pounds 711.95$